

# ALL REGULATORS OF FLAT BUNDLES ARE TORSION

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*In the Memory of Georgii Isaakovich Katz, the Teacher and the Man.*

## 1. INTRODUCTION

In our previous paper [22] we proved the Bloch conjecture on rationality of secondary characteristic classes of flat rank two vector bundles over a compact Kähler manifold, and announced the full higher dimensional generalization, which we now state.

**1.1 THEOREM. (the Bloch conjecture) Let  $X$  be a smooth complex projective variety. Let  $\rho : \pi_1(X) \rightarrow SL_n(\mathbb{C}), n \geq 2$ , be a representation of the fundamental group and let  $E_\rho$  be the corresponding holomorphic flat vector bundle over  $X$ . For  $i \geq 2$  let  $c_i \in H_{\mathcal{D}}^{2i}(X, \mathbb{Z}(i))$  be the Chern class in the Deligne cohomology group of  $X$  [10],[12]. Then  $c_i$  is a torsion class.**

I have some reason to believe that the statement may still be true for any bundle with a holomorphic connection. (c.f. [13]).

**1.2** The theorem 1.1 contrasts the rank one case, where the secondary class of the flat bundle, associated to a representation  $\rho : \pi_1(X) \rightarrow GL_1(\mathbb{C}) = \mathbb{C}^*$  is essentially the representation itself, so it may be completely arbitrary. But this freedom becomes limited when one looks at the line bundles over curves, corresponding via the Deligne-Ramakrishnan construction to elements of  $K_2(X)$ . The map  $r : K_2(X) \rightarrow H^1(X, \mathbb{C}^*)$  which appears here, is called the Bloch-Beilinson regulator map. When  $X$  is defined over a number field,  $k$ , the image of  $K_2(X_k)$  under  $r$  shows remarkable properties which were resumed in the celebrated Bloch-Beilinson conjecture [1], [20].

**1.3** The way in which we succeeded to prove the Bloch conjecture for rank two bundles in [22] was the following. Due to results, proved in succession by Bloch [2], Soulé [26], and Gillet-Soulé [15], one knows that the class  $c_i$  coincides with the image of the secondary characteristic class  $Ch_{2i-1}(\rho)$  of the flat bundle  $E(\rho)$ , lying in  $H^{2i-1}(X, \mathbb{C}/\mathbb{Z})$  and corresponding to the (torsion) Chern class  $c_i(E_\rho)$  in  $H^{2i}(X, \mathbb{Z})$  under the natural map  $H^{2i-1}(X, \mathbb{C}/\mathbb{Z}) \rightarrow H^{2i-1}(X, \mathbb{C}/\mathbb{Z}(i)) \rightarrow H_{\mathcal{D}}^{2i}(X, \mathbb{Z}(i))$ . Next, the representation  $\rho$  defines (up to homotopy) a map  $\hat{\rho} : X \rightarrow BSL_n^\delta(\mathbb{C})$ , and one knows from Beilinson, Soulé, and Dupont-Sah, that there exists a universal class  $\hat{c}_{2i-1} \in H^{2i-1}(BSL_n^\delta(\mathbb{C}), \mathbb{C}/\mathbb{Z})$ , such that  $\hat{\rho}^* \hat{c}_{2i-1} = Ch_{2i-1}(\rho)$ . An application of the Vinberg lemma shows that  $\hat{\rho}$  decomposes through  $BSL_n^\delta(\mathcal{O}_S)$ , where  $\mathcal{O}_S \subset F$  is a localized ring of integers in a number field  $F$ . With this in mind, the Theorem 1.1 follows from the following one.

**MAIN THEOREM.** *Let  $X$  be a compact Kähler manifold and let  $\rho : \pi_1(X) \rightarrow SL_n(\mathcal{O}_S)$ . Then the image under  $\hat{\rho}_*$  of  $H_{2i-1}(X, \mathbb{Z})$  in  $H_{2i-1}(BSL_n(\mathcal{O}_S), \mathbb{Z})$  is torsion.*

Indeed, the values of the universal class  $\hat{c}_i$  on the image of  $H_{2i-1}(X, \mathbb{Z})$  would then lie in  $(\mathbb{C}/\mathbb{Z})_{tors} = \mathbb{Q}/\mathbb{Z}$ .

There were two fundamental ideas, which led us to the proof of this result in the case  $i = 2$ . The first was to make use of the Borel's theorem [4] which states that the stable real cohomology  $H^*(BSL(\mathcal{O}_S), \mathbb{R})$  is multiplicatively generated by  $\sigma_k^* Vol_{2j-1}$ . Here  $\{\sigma_k\}$  is the maximal set of nonconjugate embeddings of  $F$  in  $\mathbb{C}$  (we always take  $F$  big enough so that there would not be any real embeddings). The element  $Vol_{2j-1} \in H^{2j-1}(BSL^\delta(\mathbb{C}), \mathbb{R})$ , the  $\mathbb{R}$ - part of  $\frac{1}{\sqrt{-1}} \hat{c}_i$ , is the famous Borel regulator which we called the hyperbolic volume regulator for  $j = 2$ . Recall that the  $\mathbb{R}/\mathbb{Z}$ - part of  $\hat{c}_i$  is the (universal) Cheeger-Chern-Simons class.

Secondly, we developed a general approach to regulators of flat bundles and gave a very precise geometrical recipe for their computation. This point of view results from our study in [21] and is relevant to the approach of Corlette, c.f. [8],[9].

The fundamental analytic tool which was used in [22], was the existence and rigidity for twisted harmonic maps, a theory developed by Donaldson, Corlette, Sampson, Carlson and Toledo. The key point was the use of the Sampson's degeneration theorem, which

relies heavily on the dimension restriction  $i = 2$ . So to prove the Main Theorem 1.1 in its full generality, we need to invent some new ideas.

**1.4** The strategy chosen in the present paper will be to use the refined version of the Sampson's Theorem [25] and the deformation theory of flat bundles over projective base, developed by Hitchin, Sampson, Carlson – Toledo, Goldman – Millson and Simpson, for proving the following key result. Two different proofs will be given.

**THEOREM.** *Let  $X$  be a compact Kähler manifold and let  $\rho$  be as in 1.1. Then for any  $j \geq 2$  the volume regulator  $\hat{\rho}^*(Vol_{2j-1}) \in H^{2j-1}(X, \mathbb{R})$  is zero.*

For  $j = 2$  this is proved in [22], 4.3-4.5. In the course of the proof we will rely on the geometrical description of the regulators, given in [22], 3.1-3.3.

I take the opportunity to thank all people, whose influence, personal or through their work, determined my attitude to geometry. I am deeply thankful to Michael Gromov, M. S. Narasimhan, Nigel Hitchin and Shicheng Wang. I am particularly grateful to Prof. Narasimhan for generous sharing his insight on Donaldson's invariants, which became a starting point in my thinking on secondary invariants and their relation to moduli spaces and algebraic  $K$ -theory. Also, the lectures of Prof. Hitchin on Higgs bundles during the Edinburgh Symposium in 1991 became a real clue for the approach presented here. I am equally grateful to Carlos Simpson, Max Karoubi, Hélène Esnault for interesting discussions on regulators and flat bundles, and to the Editor, Pierre Deligne, for many useful suggestions and improvements. Finally, I would like to thank Professor Ilya Rips for the most attentive and critical attitude to my work. Special thanks are due to Charlotte Herman for the excellent typing of the manuscript. This paper is dedicated to the memory of my teacher, G.I. Katz.

## 2. PROOF OF THE MAIN THEOREM.

Let  $X, \rho$  be as in 1.1. The proof of the theorem 1.1, and also theorems 1.3 and 1.4 will be divided into the following steps.

**2.1** Consider the representation variety  $V_{\Gamma}^{SL_n}$ , where  $\Gamma = \pi_1(X)$ , and find a  $\bar{\mathbb{Q}}$ -point  $\bar{\rho}$  in the component, containing  $\rho$  (c.f. [22], 2.2). Then  $Ch_{2i-1}(\bar{\rho}) = Ch_{2i-1}(\rho)$  ([7], Proposition

3.8 and [22],5.16.1). Moreover, we may assume that  $\bar{\rho}$  is defined over some  $\mathcal{O}_S$  in a number field  $F$  ([22],2.2). Relabel  $\bar{\rho}$  again by  $\rho$ .

**2.2.** Consider a continuous map  $\hat{\rho} : X \rightarrow BSL_n^\delta(\mathcal{O}_S)$ , induced by  $\rho$ . There exists a universal regulator  $\hat{c}_{2i-1}(BSL(\mathcal{O}_S), \mathbb{C}/\mathbb{Z})$  such that  $\hat{\rho}^* \hat{c}_{2i-1} = Ch_{2i-1}(\rho)$  ([1],[10],[12],[15]). We wish to prove that  $\hat{\rho}_*(H_{2i-1}(X, \mathbb{Z})) \subseteq H_{2i-1}(BSL(\mathcal{O}_S), \mathbb{Z})$  is torsion. Assuming this, we see that the value of  $\hat{c}_{2i-1} \circ \hat{\rho}_*$  on any class in  $H_{2i-1}(X, \mathbb{Z})$  will lie in  $(\mathbb{C}/\mathbb{Z})_{tors} = \mathbb{Q}/\mathbb{Z}$  as desired.

Now,  $H_*(BSL(\mathcal{O}_S), \mathbb{Z})$  is of finite type [5] and to check the claim of Theorem 1.3 it suffices to test on  $\hat{\rho}_*(H_{2i-1}(X, \mathbb{Z}))$  the generators of  $H^{2i-1}(BSL(\mathcal{O}_S), \mathbb{R})$ . The theorem of Borel [4] provides a set of multiplicative generators for  $H^*(BSL(\mathcal{O}_S), \mathbb{R})$  of the form  $\sigma_k^* Vol_{2j-1}$ , where  $Vol_{2j-1} \in H^{2j-1}(BSL(\mathbb{C}), \mathbb{R})$  is the Borel regulator and  $\sigma_k : F \rightarrow \mathbb{C}$  form a maximal set of nonconjugate embeddings. We take  $F$  big enough so that there will be no real embeddings. We claim that for all  $k, j$ ,  $\sigma_k^* Vol_{2j-1}$  vanishes on  $\hat{\rho}_*(H_{2j-1}(X, \mathbb{Z}))$ , or, equivalently,  $Vol_{2j-1}$  vanishes on  $\widehat{\sigma_k \circ \rho}_*(H_{2j-1}(X, \mathbb{Z}))$  or further,  $Vol_{2j-1}(\sigma_k \circ \rho) = 0$  in  $H^{2j-1}(X, \mathbb{R})$ . We refer to [22], for the detailed explanation of volume invariants of representations.

**2.3.** We now see that the Theorem 1.1 and the Theorem 1.3. follows from the Theorem 1.4. applied to all Galois twists  $\sigma_k \circ \rho_*$ . I call the reader's attention to the fact, that in order to prove the Theorem 1.1 even for an unitary representation (for which 1.4. trivially holds) , we need to show the vanishing of volume invariant of non-unitary representations.

So we start to prove the Theorem 1.4. saying that  $Vol_{2j-1}(\rho) = 0$ .

Denote by  $\mathcal{F}$  the associated flat  $SL_n(\mathbb{C})/SU(n)$  - bundle over  $X$ . Recall that by [22],3.3,  $Vol_{2j-1}(p)$  can be described as follows. Consider the canonical invariant  $(2j-1)$  - form  $\omega_{2j-1}$  on  $SL_n(\mathbb{C})/SU(n)$ . Identifying the tangent space to the base point  $SU(n)$  with the space  $p$  of traceless Hermitian matrices, this can be written as  $\omega_{2j-1}(A_1, \dots, A_{2j-1}) = Alt(Tr A_1 \dots A_{2j-1})$ . Next,  $\omega_{2j-1}$  lifts to a closed  $(2j-1)$  form on  $\mathcal{F}$  and using any section,  $s$ , of  $\mathcal{F}$  we pull this lifted form down to  $X$  to obtain a closed form, whose class, denoted  $Bor(\rho, \omega)$  in [22],3.2.2, gives  $Vol_{2j-1}(\rho) \in H^{2j-1}(X, \mathbb{R})$ . Since  $SL_n(\mathbb{C})/SU(n)$  is contractible, all sections are homotopic so that the resulted class is independent on the

choice of a section.

As in [22], 4.5, we now take  $s$  to be harmonic. This is possible by Donaldson - Corlette, at least if  $\rho$  is semisimple. Locally we can view  $s$  as a harmonic map  $s : X \rightarrow SL_n(\mathbb{C})/SU(n)$ , defined up to a left shift. We claim that the pull down form  $s^*\omega_{2j-1}$  is identically zero. Indeed, this will follow from the following result.

**2.4 Theorem** (Sampson [25]). Let  $x \in X$ . Identify (uniquely up to a conjugation)  $T_{s(x)}\mathcal{F}_x$  with  $p$ , where  $\mathcal{F}_x$  is a fiber over  $x$ . Then the image of  $T_x^{1,0}X$  under  $Ds \otimes \mathbb{C} : T_x^{\mathbb{C}}X \rightarrow p^{\mathbb{C}}$  is an abelian subspace of  $p^{\mathbb{C}}$ , and analogously for  $T_x^{0,1}X$ .

PROOF: For usual (untwisted) harmonic maps this is contained in [25]. The proof for harmonic sections is completely identical.

**2.5** Now decomposing  $s^*\omega_{2j-1}$  to  $(p, q)$ -components we see that  $s^*\omega_{2j-1} = \sum_{i=0}^{2j-1} c_i \nu_i \lambda_i$ , where  $c_i \in \mathbb{C}$  and for  $v_1, \dots, v_i \in T_x^{1,0}X$ ,  $\nu_i(v_1 \dots v_i) = \text{Alt}(Tr_{\mathbb{C}}\left(\left(D_x s \otimes \mathbb{C}(v_1)\right) \cdots \left(D_x s \otimes \mathbb{C}(v_i)\right)\right))$ , and similarly for  $\lambda_i$ . Now 4.4 implies immediately that  $s^*\omega_{2j-1} = 0$  for  $j \geq 2$ . In other words, if  $v_i = v'_i + v''_i$ , such that  $v'_i$  commutes, as well as  $v''_i$ , then  $\text{Alt}(Tr(v_1 \cdots v_{2j-1})) = 0$ .

**2.6 Corollary** (cf [6]). Any continuous map of a compact Kähler manifold to  $\Gamma \backslash SL_n(\mathbb{C})/SU(n)$  induces trivial map in  $H^i$  for  $i \leq m(SL_n(\mathbb{C}))$  (the Matsushima constant).

PROOF:  $H^*(SL_n(\mathbb{C})/SU(n))$  is multiplicatively generated by  $\omega_{2j-1}$  up to this range, by Matsushima (c.f. [4]).

**2.7** To deal with non-semisimple representations, consider the abelian category  $Mod(\mathbb{C}[\pi_1(X)])$  of finite-dimensional modules over  $\pi_1(X)$ . The volume invariant  $Vol_{2j-1}$  is additive on exact sequences, in other words, it defines a homomorphism  $Vol_{2j-1} : K_0(Mod) \rightarrow H^{2j-1}(X, \mathbb{R})$ . So if it vanishes on simple modules, it vanishes everywhere.

### 3. RUDIMENTS ON HIGGS BUNDLES AND THE DEFORMATION THEORY.

**3.1** Let  $X$  be a compact Kähler manifold. Consider a representation  $\rho : \pi_1(X) \rightarrow SL_n(\mathbb{C})$  and the corresponding flat vector bundle  $E_\rho$ . If  $\rho$  is unitary, equivalently, if there exists a

parallel Hermitian metric in  $E_\rho$ , then the behavior of the de Rham complex

$$0 \rightarrow \underline{\mathbb{C}} \otimes E_\rho \rightarrow C^\infty(E_\rho) \rightarrow \Omega^1(E_\rho) \rightarrow \dots$$

is completely analogous to that in the untwisted case  $\rho = 1$ . Indeed, one defines the Laplacian  $\Delta : \Omega^i(E_\rho) \rightarrow \Omega^i(E_\rho)$  using the flat connection and the whole Hodge theory applies unchanged. (In particular, for any  $i$  odd, the dimension of  $H^i(\underline{\mathbb{C}} \otimes E_\rho)$  is even and its positivity for some odd  $i$  implies  $H^2(\pi_1(X), \mathbb{Q}) \neq 0$ ).

For arbitrary representation one asks whether there exists a distinguished Hermitian metric on  $E_\rho$  such that the scene for the Hodge theory is (partially) set. The positive answer for semisimple representation was given in the remarkable series of papers by Nigel Hitchin [16], Simon Donaldson [11], Kevin Corlette [8], [9], and Carlos Simpson [27]. The metric one chooses is just a harmonic section of the associated  $SL_n(\mathbb{C})/SU(n)$  bundle. Its existence is proved by the classical nonlinear heat equation method of Eells-Sampson and the compactness of the target is replaced by the sufficient twisting of the bundle, more precisely, by the condition that the monodromy action at  $S_\infty(SL_n(\mathbb{C})/SU(n))$  does not have fixed points, so that it is energetically unprofitable for a minimizing sequence of sections to leave all compact sets.

**3.2.** Recall that any harmonic map  $f$  of a Riemannian surface  $X$  to a Riemannian manifold  $(M, g)$  gives rise to a holomorphic object - the famous holomorphic quadratic differential  $(f^*g)^{2,0} \in H^0(K^2)$ . Completely analogous, a harmonic section of the  $SL_n(\mathbb{C})/SU(n)$  - bundle above gives rise to a holomorphic object - a holomorphic section  $\theta$  of  $T_X^* \otimes \text{End} E_\rho$  for some new holomorphic structure in  $E_\rho$ . The latter had got a name “a Higgs field” in the N. J. Hitchin’s ground - breaking paper [13]. Developing the gracious theory of Narasimhan - Seshadri [19], Donaldson [11] and Uhlenbeck - Yau [29], which had established the bundle equivalence flat unitary  $\leftrightarrow$  stable with vanishing  $c_1, c_2$ , Hitchin was able to prove that any  $\theta$ -stable holomorphic bundle over (one-dimensional)  $X$  carries a “Higgs metric”  $K$  for which  $F_K = -[\theta, \theta^*]$ . Then an elementary computation shows that the perturbed connection  $\nabla_K + \theta + \theta^*$  is flat, which makes the diagram

$$\text{flat simple} \leftrightarrow \text{Higgs stable of degree zero},$$

a two-way highway. Hitchin also discovered a lot of additional geometrical structure in the moduli space of flat bundles  $\mathcal{G} = \text{Hom}(\Gamma, SL_2(\mathbb{C}))/SL_2(\mathbb{C})$  (or rather its nonsingular twisted counterpart), where  $\Gamma = \pi_1(X)$  is the surface group. In fact, the moduli space of stable bundles  $\mathcal{G} = \text{Hom}(\Gamma, SU_2)/SU_2$  sits as a real symplectic submanifold in  $\mathcal{G}$  with respect to the Goldman's symplectic structure. Moreover,  $\mathcal{G}$  is hyperkähler and with respect to one special Kähler structure, (which does not agree with the Goldman's structure) induced by the conformal structure of  $X$ ,  $\mathcal{H} \subset \mathcal{G}$  is a complex subvariety and  $\mathcal{G}$  contains a copy of  $T_{\mathbb{C}}^*(\mathcal{H}_{reg})$ . The leaves of  $T_{\mathbb{C}}^*\mathcal{H}$  over a smooth point (= a stable bundle  $E$  over  $X$ ) consists of all the Higgs fields in  $E$ . The quadratic map

$$\det : T_{\mathbb{C}}^*(\mathcal{H}) \rightarrow H^0(K^2)$$

determines a bunch of holomorphic metrics on  $\mathcal{H}$  which Poisson commute to each other. The Lagrangian subvariety  $\det^{-1}(pt)$  is identified as the Primian of that double covering of  $X$ , which makes the bundle  $E$  to be an extension of linear bundles (eigen bundles of  $\theta$ ).

**3.3.** There is a clear and obvious need to extend this beautiful picture to cover the representations of all Kähler groups. In the part of the equivalence relation this was realized by Simpson. The methods are largely the same, as far as the answer, which says

$$(*) \quad \text{flat simple} \leftrightarrow \text{Higgs stable with } c_1, c_2 = 0.$$

One remarkable feature of the key correspondence  $(*)$  is that the left side is, so to say, much more subject to variations than the right side. For instance, the field  $t \cdot \theta, t \in \mathbb{C}^*$  is again a Higgs field and we recover following Simpson [27] a  $\mathbb{C}^*$ -action in  $(*)$ . The fixed points of this action correspond precisely to variations of complex Hodge structure and the Higgs operators  $\theta(Z), Z \in T_x X$ , are all nilpotent in this case. Moreover, Simpson establishes the following absolutely magnificent theorem.

**THEOREM. (Simpson).** *Let  $X$  be smooth projective. Then any representation  $\rho : \pi_1(X) \rightarrow GL_n(\mathbb{C})$  can be smoothly deformed to a variation of a complex Hodge structure.*

Simpson gives a list of simple real Lie groups which may occur as the (real) Zariski closures of the monodromy subgroup  $\rho(\pi_1(X)) \subset GL_n(\mathbb{C})$ . These groups, called the

groups of Hodge type, are divided to two subcategories: the symmetry groups of compact symmetric Hermitian manifolds and some other groups, including two infinite series and a number of exceptional groups. By a result of Borel [3], any group of Hermitian type is really a Zariski closure of a uniform lattice. On the other hand, I don't know whether all the groups of Hodge type which are not of Hermitian type, actually occur as Zariski closures of the monodromy representations.



**4.1.** This is the list of all simple noncompact groups of Hodge type, by Simpson:

$SU(p, q)$	$Sp(n, \mathbb{R})$
$SO^*(2n)$	$E_{6(-14)}$
$SO(p, 2)$	$E_{7(-25)}$
(all of Hermitian type)	
$SO(p, 2q), q \geq 2$	$E_{8(8)}$
$Sp(p, q)$	$E_{8(-24)}$
$E_{6(2)}$	$F_{4(4)}$
$E_{7(7)}$	$F_{4(-20)}$
$E_{7(-5)}$	$G_{2(2)}$

**4.2.** For any group  $G$  from the list of 3.1. put  $K \subset G$  to be a maximal compact subgroup. Then  $\text{rank } K = \text{rank } G$  [27] and  $M = G/K$  is a symmetric space. The following table gives the Poincaré polynomials of the dual compact symmetric space  $\hat{M}$  (c.f. Fomenko [14]).

Group $G$	Subgroup $K$	$G/K = M$	$\hat{M}$	$P_{\hat{M}}(t)$
$SU(p, q)$	$SU(p) \times SU(q)$	$\frac{SU(p, q)}{S(U(p) \times U(q))}$	$\frac{SU(p+q)}{S(U(p) \times U(q))}$	$\frac{(1-t^{2(q+1)}) \dots (1-t^{2(q+p)})}{(1-t^2) \dots (1-t^{2p})}$
$SO^*(2n)$	$U(n)$	$\frac{SO^*(2n)}{U(n)}$	$\frac{SO(2n)}{U(n)}$	$(1+t^2) \dots (1+t^{2n-2})$
$SO(p, 2), p \text{ odd}$	$SO(2) \times SO(p)$	$\frac{SO(p, 2)}{SO(2) \times SO(p)}$	$\frac{SO(p+2)}{SO(2) \times SO(p)}$	$1 + t^2 + \dots + t^{2p-4}$
$SO(p, 2), p \text{ even}$	$SO(2) \times SO(p)$	$\frac{SO(p, 2)}{SO(2) \times SO(p)}$	$\frac{SO(p+2)}{SO(2) \times SO(p)}$	$(1 + t^2 + \dots + t^{p-2}) \times$ $\times (1 + t^{p-2})$
$Sp(n, \mathbb{R})$	$SU(n)$	$\frac{Sp(n, \mathbb{R})}{SU(n)}$	$\frac{Sp(n)}{SU(n)}$	$(1 + t^2) \dots (1 + t^{2n})$
$E_{6(-14)}$	$SO(10) \times SO(2)$	$\frac{E_{6(-14)}}{SO(10) \times SO(2)}$	$\frac{E_{6(-78)}}{SO(10) \times SO(2)}$	$(1 + t^2 + \dots + t^{16}) \times$ $\times (1 + t^8 + t^{16})$
$E_{7(-25)}$	$E_6 \times SO(2)$	$\frac{E_{7(-25)}}{E_6 \times SO(2)}$	$\frac{E_{7(-133)}}{E_6 \times SO(2)}$	$(1 + t^2 + \dots + t^{26}) \times$ $\times (1 + t^{10})(1 + t^{18})$
$SO(p, 2q), p \text{ odd}$	$SO(p) \times SO(2q)$	$\frac{SO(p, 2q)}{SO(p) \times SO(2q)}$	$\frac{SO(p+2q)}{SO(p) \times SO(2q)}$	$\frac{(1-t^{2(p+1)}) \dots (1-t^{2(p+2q-1)})}{(1-t^4) \dots (1-t^{4(q-1)})(1-t^{2q})}$
$SO(p, 2q), p \text{ even}$	$SO(p) \times SO(2q)$	$\frac{SO(p, 2q)}{(SO(p) \times SO(2q))}$	$\frac{SO(p+2q)}{SO(p) \times SO(2q)}$	$\frac{(1-t^{2p}) \dots (1-t^{2(p+2q-2)})}{(1-t^4) \dots (1-t^{4(q-1)})}$
$Sp(p, q)$	$Sp(p) \times Sp(q)$	$\frac{Sp(p, q)}{Sp(p) \times Sp(q)}$	$\frac{Sp(p+q)}{Sp(p) \times Sp(q)}$	$\frac{(1-t^{4(p+1)}) \dots (1-t^{4(p+q)})}{(1-t^4) \dots (1-t^{4p})}$

$E_{6(2)}$	$SU(6) \times SU(2)$	$\frac{E_{6(2)}}{SU(6) \times SU(2)}$	$\frac{E_{6(-78)}}{SU(6) \times SU(2)}$	$(1 + t^4 + \dots + t^{20}) \times$ $\times (1 + t^6 + t^{12})(1 + t^8)$
$E_{7(7)}$	$SU(8)$	$\frac{E_{7(7)}}{SU(8)}$	$\frac{E_{7(-133)}}{SU(8)}$	$(1 + t^6 + \dots + t^{30}) \times$ $\times (1 + t^8 + t^{16}) \times$ $\times (1 + t^{10})(1 + t^{14})$
$E_{7(-5)}$	$SO(12) \times SU(2)$	$\frac{E_{7(-5)}}{SO(12) \times SU(2)}$	$\frac{E_{7(-133)}}{SO(12) \times SU(2)}$	$(1 + t^4 + \dots + t^{24}) \times$ $\times (1 + t^8 + t^{16}) \times$ $\times (1 + t^{12} + t^{24})$
$E_{8(8)}$	$SO(16)$	$\frac{E_{8(8)}}{SO(16)}$	$\frac{E_{8(-248)}}{SO(16)}$	$(1 + t^8 + \dots + t^{32}) \times$ $\times (1 + t^{12} + t^{24}) \times$ $\times (1 + t^{16} + t^{32}) \times$ $\times (1 + t^{20} + t^{40})$
$E_{8(-24)}$	$E_7 \times SU(2)$	$\frac{E_{8(-24)}}{E_7 \times SU(2)}$	$\frac{E_{8(-248)}}{E_7 \times SU(2)}$	$(1 + t^4 + \dots + t^{36}) \times$ $\times (1 + t^{12} + t^{24} + t^{36}) \times$ $\times (1 + t^{20} + t^{40})$
$F_{4(4)}$	$Sp(3) \times SU(2)$	$\frac{F_{4(4)}}{Sp(3) \times SU(2)}$	$\frac{F_4}{Sp(3) \times SU(2)}$	$(1 + t^4 + \dots + t^{20}) \times$ $\times (1 + t^8)$
$F_{4(-20)}$	$Spin(9)$	$\frac{F_{4(-20)}}{Spin(9)}$	$\frac{F_4}{Spin(9)}$	$1 + t^8 + t^{16}$
$G_{2(2)}$	$SO(4)$	$\frac{G_{2(2)}}{SO(4)}$	$\frac{G_2}{SO(4)}$	$1 + t^4 + t^8$

We deduce the following proposition.

**4.3 PROPOSITION.** *For any simple noncompact group  $G$  of Hodge type, the odd dimensional cohomology  $H^{\text{odd}}(\hat{M}) = 0$ . Consequently, any  $G$  - invariant form in  $G/K$  is of even dimension.*

**PROOF:** The first statement follows from the inspection of 3.2. The second is equivalent to it by duality (see Borel [4]).

**4.4. REMARK:** The reader may wish to ask is there is a natural explanation for the phenomenon, recovered in 3.3. We refer to our paper [23] for the relevant computation.

**4.5** We will give now a different proof of Theorem 1.4. Using the Simpson's fundamental result we deform  $\rho$  to a variation of Hodge structure, which we call  $\mu$ . By [7],  $\text{Vol}_{2j-1}(\mu) =$

$Vol_{2j-1}(\rho)$ . So it is enough to show that  $Vol_{2j-1}(\mu) = 0$ . Let  $G \subset SL_n(\mathbb{C})$  be the Zariski closure of  $\mu(\pi_1(X))$ , then  $G = \tilde{K} \times G_1 \times \dots \times G_n$ , where  $\tilde{K}$  is compact and  $G_i$  are from the list of 3.1. Choose maximal compact subgroups  $K_i \subset G_i$  and  $K \subset SL_n(\mathbb{C})$  such that  $K \cap G_i = K_i$ , and  $\tilde{K} \subset K$ . Then we have a totally geodesic embedding

$$G/K \subset SL_n(\mathbb{C})/SU(n)$$

Now, let  $\omega \in \Omega^{2j-1}(SL_n(\mathbb{C}), SU(n))$  be an invariant form such that in terms of [22],  $Vol_{2j-1}(\mu) = Bor(\omega, \mu)$ . Restricting an  $G/K$  we get by naturality  $Vol_{2j-1}(\mu) = Bor(\omega|_{G/K}, \mu)$ . However, the form  $\omega|_{G/K}$  is an invariant form of odd dimension on  $G/K = \coprod G_i/K_i$ , which should be zero by 3.3. Hence  $Vol_{2j-1}(\mu) = 0$ , which proves theorem 1.4 and hence also theorem 1.3 and the Theorem 1.1.

## 5. REPRESENTATIONS IN LOW DIMENSIONS

**5.1.** The aim of this section is to give a spirit of what kind of behavior may be expected from representations of Kähler groups in  $SL_n(\mathbb{C})$ , for  $n$  small ( $n \leq 3$ ).

By the fundamental result of Simpson, any representation may be deformed to a variation of Hodge structure, so one should begin to study these first.

Let  $X$  be a Kähler manifold and let  $\rho : \pi_1(X) \rightarrow SL_2(\mathbb{C})$  is a semisimple variation of Hodge structure. The Zariski closure of  $\rho(\pi_1(X))$  in  $SL_2(\mathbb{C})$  is either a subgroup of  $SU(2)$  or  $SL_2(\mathbb{R})$  or  $PSU(1, 1) \approx SL_2(\mathbb{R})$ . We have nothing to say in the first case. Consider therefore a representation in  $SU(1, 1)$ , which we relabel  $\rho$ .

Consider the flat bundle  $E_\rho$ . By Simpson's theory, the corresponding Higgs bundle decomposes as  $L \oplus M$  and the Higgs field  $\theta$  lies in  $\Omega^1(X, Hom(L, M))$ . Observe that  $M = L^{-1}$  since there exists a parallel symplectic form in  $E_\rho$ . So  $\theta \in \Omega^1(X, M^2)$ . Next, since  $E_\rho$  is flat we get  $0 = c_2(E_\rho) = -c_1^2(M)$ .

So we get the following data:

- (i) a line bundle  $M$  with  $c_1^2(M) = 0$  and  $\deg M < 0$
- (ii) a form  $\theta \in \Omega^1(X, M^2)$ .

Conversely, given such data we construct a representation at  $\pi_1(X)$  in  $SL_2(\mathbb{R})$  by the Simpson's theory. Observe that a one-form with values in a line bundle gives rise to

a codimension one holomorphic foliation of  $X$  (with singularities). In fact, Simpson and Corlette proved that any nonrigid representation decomposes through an orbifold curve.

**5.2 EXAMPLE:** Suppose  $X$  is a surface with  $H^{1,1}(X) \cap H^2(X, \mathbb{Q})$  one-dimensional [24]. This is a generic condition. Then by (i) we have  $c_1(M) = 0$  since the intersection form is positively defined in the (one-dimensional) Hodge cycles subspace. But this is impossible, since  $\deg M$  should be negative. So such surfaces do not admit nontrivial homomorphisms  $\rho : \pi_1(X) \rightarrow SL_2(\mathbb{R})$ .

**5.3 EXAMPLE:** Suppose  $X$  is a Hilbert modular surface, i.e.  $X = \mathcal{H}^2 \times \mathcal{H}^2 / \Gamma$ , where  $\Gamma$  is a uniform irreducible lattice in  $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ . Obviously  $TX$  splits as  $TX = L_1 \oplus L_2$ . Consider the line bundle  $L_i^{1/2}, i = 1, 2$ . It exists and may be looked at as a spinor bundle, corresponding to a flat  $\mathcal{H}^2$ -bundle over  $X$ , associated to defining representations  $\rho_{1,2} : \Gamma \rightarrow SL_2(\mathbb{R})$ . (cf. [11]). As such it carries the canonical Hermitian metric with nonpositive curvature form. It is convenient to consider the local  $\mathcal{H}^2$ -system over  $X$ , associated to  $\rho_i, i = 1, 2$ , along with the canonical (developing) section. Then  $L_i^{1/2}$  is just  $K^{-1/2}$ , where  $K$  is the canonical bundle of the fiber, looked at as a Riemannian surface. In particular,  $(c_1(L_i^{1/2}))^2 = 0$  and  $\deg L_i^{1/2} < 0$ . Since  $\text{rank}(SL_2(\mathbb{R}) \times SL_2(\mathbb{R})) = 2, \Gamma$  is arithmetic, so  $X$  is projective. The form  $\theta_i \in \Omega^1(X, L_i)$  is just the projection  $L_1 \oplus L_2 \rightarrow L_i$ . The two representations  $\rho_i : \Gamma \rightarrow SL_2(\mathbb{R})$  are rigid by Margulis.

**5.4.** Now we turn to the secondary class of  $\rho$ . This is just  $\lambda((c_1^{Chow}(M))^2)$ , where  $\lambda : Ch_0^i(X) \rightarrow H_{\mathcal{D}}^{2i}(X, \mathbb{Z}(i))$  is the Deligne cycle class map and is torsion by [22], Main Theorem, and by 1.1. If we remove the condition (ii), we may ask the following question:

**QUESTION:**

Suppose  $D$  is a divisor on  $X$  such that  $\deg D < 0$  and  $[D]^2 = 0$  in singular cohomology. Consider  $([D]^{Chow})^2 \in Ch_0^2(X)$ . Is  $\lambda([D]^{Chow})^2$  always torsion?

We see that the answer is yes, if  $T^*(X) \otimes \mathcal{L}^{\otimes 2}(D)$  admits a holomorphic section.

**5.5.** Now suppose  $\Gamma$  is an arithmetic lattice in  $SU(1, 2)$  and  $X = B^2 / \Gamma$  is a hyperbolic surface. Let  $\rho : \Gamma \rightarrow SU(1, 2) \rightarrow SL(3, \mathbb{C})$  be the defining representation and let  $E_\rho$  be the associated rank 3 flat vector bundle.

We wish first to understand the underlying holomorphic structure of  $E_\rho$  (induced by the flat connection). Realize  $B^2$  as the quotient of the cone  $Q = (|x_1|^2 - |x_2|^2 - |x_3|^2 > 0)$  by the action of  $\mathbb{C}^*$ . We will have then the canonically defined  $SU(2, 1)$  - invariant tautological line bundle  $L$  over  $Q/\mathbb{C}^* = B^2$  (the analogue of  $\mathcal{O}(-1)$  over  $\mathbb{P}^2$ ) and an  $SU(2, 1)$  - equivariant embedding  $y : L \rightarrow B^n \times \mathbb{C}^{n+1}$ . Descending to  $X = B^2/\Gamma$ , we get a “hyperplane” bundle, denoted also by  $L$ , and the embedding  $y : L \rightarrow E_\rho$ . Moreover, the quotient  $\mathbb{C}^{n+1}/\varphi(L)$  is  $T(B^n) \otimes L$ , so we get an exact sequence

$$0 \rightarrow L \rightarrow E_\rho \rightarrow L \otimes TX \rightarrow 0$$

. The extension class  $\varphi \in H^1(\text{Hom}(L \otimes TX, L)) = H^1(\Omega^1)$  is proportional to the Kähler class of  $X$ . The corresponding Higgs bundle is a direct sum  $L \oplus L \otimes TX$  and  $\theta$  is given by a matrix  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ . Next,  $c_2^{\text{Chow}}(E_\rho)$  is just  $c_2^{\text{Chow}}(X) - \frac{1}{3}(c_1^{\text{Chow}}(X))^2$ . So we get a rational class  $B_\rho = \lambda(c_2^{\text{Chow}}(X) - \frac{1}{3}(c_1^{\text{Chow}}(X))^2) \in H^3(X, \mathbb{Q}/\mathbb{Z})$ . Observe that  $H^3(X, \mathbb{Q}/\mathbb{Z})$  may be nontrivial in many known cases, but it seems difficult to compute  $B_\rho$ .

**5.6** We will use the description of 5.5 to give a proof of a version of the following famous theorem of Yau.

**THEOREM.** *Let  $X$  be a compact Kähler surface such that i)  $\Omega^2(X)$  is positive and ii)  $c_1^2(X) = 3c_2(X)$ .*

*Then  $X$  is a ball quotient.*

We will replace (i) by a weaker assumption that  $\deg(\Omega^2(X)) = -c_1(X)\omega > 0$  but demand  $TX$  to be stable (e.g. there are no holomorphic foliations on  $X$ ). We also make a technical assumption that  $\frac{1}{3}c_1(X) \in H^2(X, \mathbb{Z})$ . Form a Higgs bundle  $L \oplus L \otimes TX$  where  $L^{\otimes 3} = \Omega^2(X)$ , and  $\theta$  is  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ . We claim it is stable. In fact, any  $\theta$  - invariant subbundle is a subbundle of  $L \otimes TX$ , say  $L \otimes E$  where  $E \subset TX$ . Compute  $\deg(L \otimes E) = \deg L \cdot \text{rank } E + \deg E = -\frac{1}{3} \deg TX \cdot \text{rank } E + \deg E$ , so  $\deg(L \otimes E) < 0$  is equivalent to  $\frac{\deg E}{\text{rank } E} < \frac{1}{3} \deg TX$  which is weaker than stability of  $TX$ . Since  $c_1(L \oplus L \otimes TX) = 0$  and  $c_2(L \oplus L \otimes TX) = 0$ , we come to a variation of Hodge structure, whose monodromy should be a Hodge type subgroup of  $SL_3(\mathbb{C})$ . It is easy to see that there is the only opportunity of  $SU(2, 1)$ . So we get a representation  $\rho : \pi_1(X) \rightarrow SU(2, 1)$  and  $E_\rho$  corresponds to  $L \oplus L \otimes TX$ .

Next, find a harmonic metric in  $E_\rho$ . This is a harmonic section of the associated  $SU(2,1)/SU(2) = B^2$  - bundle over  $X$ , say,  $s$ . By Sampson-Corlette - Carlson - Toledo, we have that either  $\text{Ker}(Ds|_x)$  is a nonzero complex subspace of  $T_x X$ , or  $s$  is holomorphic. But the field  $\theta$  is just  $(Ds \otimes \mathbb{C})^{1,0}$ , and viewed as a map  $\theta_x : T_x X^{1,0} \rightarrow \text{End}(L \oplus L \otimes TX)$  is of maximal rank everywhere. This rules out the first possibility, so  $s$  is holomorphic, and the same argument shows that  $\text{rank}_{\mathbb{C}} Ds = 2$  everywhere, so we can pull the metric of  $B^2$  down to  $X$  using  $s$  and hence  $X$  has a hyperbolic metric.

This type of argument was first used in [21] to prove the Goldman's theorem. Observe that for surfaces of general type the condition (i) is 5.6 can be removed by Bogomolov-Miyaoka.

**5.7** We wish to find a similar characterization of Hilbert modular surfaces. Recall that any Kähler manifold diffeomorphic to such surface is actually bi-holomorphic to it. We suggest the following purely geometrical description.

**THEOREM.** *Let  $X$  be a compact Kähler surface such that i) The signature  $\sigma(X) = 0$  and  $\chi(X) \neq 0$  ii) There exist two transversal holomorphic foliations on  $X$  with nonpositive normal bundles.*

*Then  $X$  is biholomorphic to a Hilbert modular surface.*

**PROOF:** We have  $TX = L_1 \oplus L_2$  and the both  $L_i$  are nonpositive. Compute  $0 = p_1(TX) = p_1(L_1) + p_1(L_2) = c_1^2(L_1) + c_1^2(L_2)$ , so  $c^2(L_1) = c_1^2(L_2) = 0$ . Moreover,  $\chi(X) = c_2(X) = c_1(L_1)c_1(L_2) \neq 0$ , so  $c_1(L_i) \neq 0$ . Since  $c_1(L_i)$  lies in the Hodge cycle subspace of  $H^{1,1}$ , we have by the Hodge index theorem  $\deg L_i = c_1(L_i)\omega \neq 0$  and since  $L_i$  is nonpositive,  $\deg L_i < 0$ . We will make a technical assumption  $c_1(L_i)$  is even, i.e.  $\frac{1}{2}c_1(L_i) \in H^2(X, \mathbb{Z})$ . Then  $L_i = M_i^2$  for some line bundle  $M_i$  and by 5.1 we get two representations  $\rho_{1,2} : \pi_1(X) \rightarrow SL_2(\mathbb{R})$ . Let  $s_i$  be the harmonic section of the corresponding that  $\mathcal{H}^2$  - bundle, then  $L_1 = \text{Ker}(Ds_2)$  and  $L_2 = \text{Ker}(Ds_1)$  so  $(s_1, s_2)$  is of maximal rank everywhere. This imposes the locally homogeneous structure on  $X$ , as above.

**5.8** We close with applications of Higgs theory to higher Milnor inequalities, referring to [21] for a general survey. Let first  $X$  be a compact Riemann surface of genus  $g > 1$  and let  $\rho : \pi_1(X) \rightarrow SU(n, 1)$  be a representation. The classifying space  $BSU^\delta(\infty, 1)$  carries a

special element  $\hat{c} \in H^2(BSU^\delta(\infty, 1))$ , introduced and studied by Morita [18], Toledo [28], Corlette [9] and the author [21]. The pull-back  $\hat{p}^*c \in H^2(X, \mathbb{R})$  is a secondary class of  $X$ , and the number  $(\hat{p}^*c, [X])$  is called the degree of  $\rho$ , denoted  $\deg \rho$ . Then we have the following beautiful result.

**THEOREM.** (*Milnor [17], Hitchin [16] ( $n = 1$ ), Toledo [28]*). For any  $X, \rho$ , we have  $|\deg \rho| \leq g - 1$ .

**PROOF:** As in [19], we may suppose  $\rho$  to be irreducible. Let  $\mathcal{F}$  be the associated flat  $B^n$ -bundle over  $X$  and let  $s$  be a harmonic section. Recall that there is a natural  $SU(n, 1)$ -equivariant line bundle  $L$  over  $B^n$ , equipped with the canonical connection. Let  $N$  be the pull-back bundle over  $X$  by  $s$  with the holomorphic structure, induced by the pull-back connection. Let  $M$  be the pull-back of the tangent bundle to  $B^n$ , again with the Levi-Civita connection and the induced holomorphic structure (over  $X$ ). The flat bundle  $E_\rho$ , as a complex bundle, is an extension of  $M \otimes N$  by  $N$ , whereas the corresponding Higgs bundle is  $N \oplus M \oplus N$  and the  $\theta$ -field is given by  $\begin{pmatrix} 0 & * \\ (Ds \otimes \mathbb{C})^{1,0} & 0 \end{pmatrix}$ . Moreover, the degree  $\deg \rho$  is just  $\deg N$ .

Since  $X$  is a curve, the image of  $N$  by  $\theta$  extends to a holomorphic subbundle  $P$  of  $M \otimes N$ , so that  $0 \neq \theta \in H^0(\Omega^1 \otimes \text{Hom}(N, P))$  (the case  $\theta = 0$  is trivial). Hence  $\deg P \geq \deg N - (2g - 2)$ . The subbundle  $N \oplus P$  is  $\theta$ -invariant, so by the Higgs stability,  $\deg P + \deg N < 0$  which means  $\deg N < g - 1$ , if  $n > 1$ . Changing the orientation on  $X$ , we see that also  $-\deg \rho < g - 1$ , so  $|\deg \rho| < g - 1$ . The equality, for  $\rho$  irreducible is possible only if  $n = 1$  and  $\Omega^1 \otimes \text{Hom}(N, P)$  trivial, which gives the standard Higgs bundle  $K^{+1/2} \oplus K^{-1/2}, \theta = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$ , corresponding to the embedding  $\pi_1(X) \rightarrow SU(1, 2) \rightarrow SU(n, 1)$ , by Hitchin [16].

**5.9** The case of higher-dimensional  $X$  is conceptually simpler because of Siu's superrigidity, which forces  $s$  to be holomorphic. We just state the corresponding inequalities, leaving the proofs to the reader.

**THEOREM.** Let  $X = B^m/\Gamma$  be a ball quotient and let  $\rho : \Gamma \rightarrow SU(n, 1)$  be a representation. Then  $\left| (\hat{p}^*c \cdot \omega^{m-1}, [X]) \right| \leq (\omega^m, [X]) = \text{Vol} X$ .

Moreover [9],  $\left| \left( \hat{p}^* c \right)^m, [X] \right| \leq Vol X$ .

It is remarkable that one can prove the parallel statement in real hyperbolic geometry, namely, that for any representation  $\rho : \pi_1(X) \rightarrow PSO(n, 1)$  of the fundamental group of a compact hyperbolic manifold, one gets  $Vol(\rho) \leq Vol(X)$ . We refer to [22] for the details.

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